Cosmology (PHYSDSE05: Astrophysics and Cosmology)
Problem Set-2

1. **The Big Rip Hypothesis and Cosmic Doomsday**
Assume dark energy to be a fluid with $w = -2$ (equation of state parameter). Show that the energy density of such a fluid increases with time. Dark energy models with $w < -1$ are called phantom energy. Read the paper by Caldwell et al. (2004) on phantom energy and cosmic doomsday. Show that for 'phantom' dark energy models, the universe eventually ends up in a Big Rip (i.e., the scale factor blows up in a finite time scale). Using the current value of the cosmological parameters show that for $w = -2$ the Universe ends up in a Big Rip in the next 11 Gyrs. Note that the possibility of a phantom energy has been almost ruled out by current data. However, these models got popular in the pre-WMAP era.

2. **Greatest order-of-magnitude error in Theoretical Physics**.
A standard approach for both classical and quantum fields is to imagine the field to be a bunch of point particles connected by small strings and take the limit when the number of particles go to infinity and the space between them goes to zero. Thus doing a normal mode analysis of both quantum and classical fields using the Fourier approach is a standard technique. This analysis gives us the allowed modes of vibration.
a) Consider the electromagnetic field inside a box of volume L^3. Just like the vibrations of strings (or particle in a box problem) you can find the allowed modes. The only difference will be here $P = E/c$. The generalization to 3 dimensions would be simple. There would be 3 quantum numbers in 3 dimensions. You will see that roughly the allowed energy of each mode would be hcn/L, where n is the mode number.
b) Now each of this vibrating mode can be imagined as a harmonic oscillator with a zero-point energy. But a field will have infinite number of oscillators. As a result when we will sum the zero-point energy of all of these modes we will get infinite energy of the vacuum.
c) That is embarrassing! So people tried to find out a cut-off scale. Let us say that the cut-off scale be the Planck scale (l_p) below which quantum gravity becomes important. So the wavelength of the maximum mode would be the planck scale (i.e., wavelength is minimum but the quantum number is maximized).
d) Using the techniques that you have learned in your statistical mechanics class, if you now integrate the result in part a) for all modes (i.e n) from 0 to the Planck scale using spherical shell i.e assuming $4\pi n^2 dn$ be the number of modes in volume dV, you will see that the energy density can be written as $U \approx hc/l_p^4$. Using the numerical value of the Planck length and other constants show that this energy density is roughly 10^{115} ergs/c.c
e) Compare the energy density in part d) with the energy density of dark energy as measured

by Planck. Do you see the discrepancy? The "greatest order of magnitude error" indeed.

3. **Equation of State of Dark Energy**
Assume dark energy to be a substance with equation of state w. Consider w to be constant (non evolving with time) with value $\leq -1/3$. Assuming a flat dark energy +dark matter universe, find the luminosity distance as a function of redshift and w. You need to evaluate the integral numerically. At what redshift do you find the differences in the luminosity distances for w $= -2/3$, w $= -1/2$, and w $= -1$. This is how cosmologists try to measure the equation of state (w) of dark energy from luminosity distances of high redshift supernova. There are of course models of dark energy where we have time-varying equations of state.

4. **Einstein's Greatest Blunder**
a) Write down the Friedmann equation for a universe with non-relativistic matter, curvature and cosmological constant. What are the conditions for a static universe?
b) From part a) find out the relation between ρ_Λ and ρ_m for a static universe where ρ_Λ and ρ_m are energy densities of the cosmological constant and matter, respectively. Show from b) and c) that such a universe will be positively curved.
c) Now slightly perturb the expansion parameter from its constant value and rewrite the Friedmann equation. Keeping second order terms in the expansion parameter (δa) show that the static universe solution is exponentially unstable. This is the reason for which Einstein was unhappy with the cosmological constant and later called it his *greatest blunder*.

5. **The horizon size at last scattering**
Calculate the horizon scale at the surface of last scattering (z=1100). Now calculate the angular diameter distance to the surface of last scattering. From these two scales calculate the angular scale in degrees of the size of the horizon at last scattering.

6. **Neutrino decoupling in the early Universe**
In Fermi's effective field theory of weak interactions, the weak cross-section (σ_w) goes as $G_F^2 E^2$, where G_F and E are the coupling constant and the energy scale, respectively. Show that, in the early Universe, for the neutrino-electron scattering, Γ_w/H goes as T^3, where Γ_w and H are neutrino-electron scattering rate and Hubble constant, respectively.