Problem Set 4: Central Force

1. **Properties of Central Force Orbits:**
a) Show that the eccentricity of a planet's orbit is given by:
$\epsilon = \frac{\mathrm{V_{max}} - \mathrm{V_{min}}}{\mathrm{V_{max}} + \mathrm{V_{min}}}$, where $\mathrm{V_{max}}$ and $\mathrm{V_{min}}$ are the maximum and minimum speed, respectively, of the planet in its orbit.
b) Show that the total energy of a planet of mass m is given by:
$\mathrm{E} = -\frac{\mathrm{GMm}}{2\mathrm{a}}$, where a is the semi-major axis of its orbit around a star of mass M.
c) Using the above, show that the speed of the planet is given by:
$\mathrm{v}^2 = \mathrm{GM}(\frac{2}{\mathrm{r}} - \frac{1}{\mathrm{a}})$, where r is its distance from the star.

2. **Finding the Nature of Central Force from a Given Orbit Equation:**
a) Find the force law for a central force field if the corresponding orbit of a particle is given by $\mathrm{r} = \mathrm{A}e^{\mathrm{a}\theta}$, where A and a are constants.
b) Find $U(r)$, as a function of angular momentum l, which is consistent with the above orbit. Note that we are looking for U and not U_{eff}.
c) Find the total energy of the particle in this orbit.

3. **Correction to Kepler's 3rd Law:**
(a) Using elementary Newtonian mechanics find the period of a mass m_1 in a circular orbit of radius r around a *fixed* mass m_2.
(b) Using the reduced-mass formalism, find the corresponding period for the case that m_2 is not fixed and the masses move around their CM being a constant distance r apart. Discuss the limit of this result if $m_2 >> m_1$.
(c) What would be the orbital period if the earth were replaced by a star of mass equal to the solar mass, in a circular orbit, with the distance between the sun and star equal to the present earth-sun distance?

4. **Geometry of Ellipses in Polar and Cartesian Coordinates:**
It was shown in class that any Kepler orbit can be written in the form $r(\phi) = c/(1 + \epsilon cos\phi)$, where $c > 0$ and $\epsilon \geq 0$. Show that the equation can be written in Cartesian coordinates as the standard equation of ellipse, parabola, and hyperbola depending on the value of ϵ.

5. **Stability of Circular Orbit:**
Consider a particle of reduced mass μ orbiting in a central force with $\mathrm{U} = \mathrm{kr^n}$ where $\mathrm{kn} > 0$.
a) Explain what the condition $\mathrm{kn} > 0$ tells us about the force. Sketch the effective potential energy ($\mathrm{U_{eff}}$) for the case n=2, -1, and -3.
b) Find the radius at which the particle (with a given angular momentum l) can orbit at a fixed

radius. For what value of n is this circular orbit stable? Do your sketch confirm this conclusion?
c) For the stable case, show that the period of small oscillations about the circular orbit is $t_{osc} = t_{orb}/\sqrt{n+2}$. For what property of n, will this orbit be closed?